An Analysis for the Compressible Stokes Equations by First-Order System of Least-Squares Finite Element Method

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This article applies the first-order system least-squares (fosls) finite element method developed by Cai, Manteuffel and McCormick to the compressible Stokes equations. By introducing a new dependent velocity flux variable, we recast the compressible Stokes equations as a first-order system. Then it is shown that the ellipticity and continuity hold for the least-squares functionals employing the mixture of $H^{-1}$ and $L^2$, so that the fosls finite element methods yield best approximations for the velocity flux and velocity. © 2001 John Wiley & Sons, Inc. Numer Methods Partial Differential Eq 17: 689–699, 2001

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I. INTRODUCTION

Let $\Omega$ be a bounded, open domain in $\mathbb{R}^n (n = 2, 3)$ with Lipschitz boundary $\partial \Omega$. Consider the system of equations of the stationary compressible Stokes equations with zero boundary conditions (cf. [1]) for the velocity $u = (u_1, \cdots, u_n)^t$ and pressure $p$ as follows:

$$\begin{align*}
-\mu \Delta u + \rho(\beta \cdot \nabla)u + \nabla p &= f, \quad \text{in } \Omega, \\
\nabla \cdot u + \kappa \beta \cdot \nabla p &= g, \quad \text{in } \Omega, \\
\nabla \cdot u &= 0, \quad \text{on } \partial \Omega,
\end{align*}$$

(1.1)

where the symbols $\Delta, \nabla$, and $\nabla \cdot$ stand for the Laplacian, gradient, and divergence operators, respectively ($\Delta u$ is the vector of components $\Delta u_i$); $\beta = (\beta_1, \cdots, \beta_n)^t$ and $P$ are given $C^1$ functions describing the given ambient flow; $\rho = \rho(P)$ is a given positive increasing function as a function of pressure; $\rho' = \frac{dp}{dp}$, $\kappa = \frac{\mu}{\rho}$; the number $\mu > 0$ is a viscous constant; $f$ is a given...
vector function. We further assume that

\[ \int_{\Omega} p \, d\Omega = 0 \]

and \( \beta = 0 \) on \( \partial \Omega \), \( \rho \), and \( \kappa \) are bounded.

The equations (1.1) are obtained by linearizing the barotropic Navier-Stokes equations around the ambient flows \( \beta \) and \( P \), neglecting the bulk viscosity constant term and the zero-order terms (see [1]). The traditional numerical approach is to use a mixed finite element method for these equations (1.1)(see [1]). The least-squares approach for second and higher order problems was developed in [2], [3], and [4], for example, and it was developed for the Stokes and Navier-Stokes equations in [5]. The analysis for the fosls methods applied to the incompressible Stokes equations by introducing a new dependent variable had been carried out by several authors (see, for example, [6] and [7] and [8]-[10]). These methods are well known as possessing the freedom of choosing finite element spaces, incorporating additional equations, and imposing additional boundary conditions as long as the system is consistent. The motivation of this article is to adopt such benefits to the compressible Stokes equations (1.1). For these, we reformulate the compressible Stokes equations as a first-order system derived in terms of an additional velocity flux. For the systems proposed in this article, the least-squares functionals are defined using \( L^2 \) and \( H^{-1} \) norms or a mixture of \( L^2 \), \( H^{-1} \) norms and integrals. Recently, a least-squares approach for the compressible Stokes equations reported in [11] did not focus on the auxiliary variable velocity flux, in which one may lose some accuracy to approximate the auxiliary flux variable from the velocity variable by numerical differentiation. Compared with the works in [11], the fosls formulations in this article give the way to approximate the flux variable directly with the best accuracy as the velocity variable has. With the assumption that the viscosity constant \( \mu \) is sufficiently large, we show that the resulting functionals are equivalent to a product norm \( \| U \|^2 + \| u \|^2 + \| p \|^2 \) following ideas in [10] (see the notation \( \| \cdot \|_\beta \) in section 2). The \( H^{-1} \) fosls approaches enable us to provide a possible optimal convergence rate, which can be a contrast to a mixed finite element method, from the convergence estimate (see Theorem 4.1). This may be one of the benefits for employing the fosls approach to the compressible Stokes equations (1.1). We also discuss the computable functional and provide its error estimate.

This article proceeds as follows. In section 2, the notations, definitions, and first-order system formulations are introduced. In section 3, the least-squares functionals are provided, and coercivity results are derived. In section 4, using these coercivity results, we note that fosls formulation can lead to the optimal convergence rate, and we establish discretization error estimates with respect to order of approximation.

II. FOSLS FORMULATION

For convenience, we will adopt the notations introduced in [12] and put the necessary definitions in this section. A new independent variable related to the \( n^2 \)-vector function of gradients of the displacement vectors, \( u_i, i = 1, \cdots, n \) will be given. For a given \( u = (u_1, \cdots, u_n)^t \), the gradient is \( \nabla u = (\partial_1 u_1, \partial_2 u_1, \partial_3 u_1, \cdots, \partial_2 u_3, \partial_3 u_3)^t \).

For a function with vector components \( U = (U_1, U_2, \cdots, U_n)^t \), the divergence is defined as

\[ \nabla \cdot U = (\nabla \cdot U_1, \nabla \cdot U_2, \cdots, \nabla \cdot U_n)^t. \]
The inner products and norms on the block column vector functions are defined in the natural componentwise way, for example:

\[ \| U \|_2^2 = \sum_{i=1}^{n} \| U_i \|_2^2 \quad \text{and} \quad \left| \int_{\Omega} U \, d\Omega \right|^2 = \sum_{i=1}^{n} \left| \int_{\Omega} U_i \, d\Omega \right|^2. \]

We use standard notations and definitions for the Sobolev spaces \( H^s(\Omega) \), associated inner products \( \langle \cdot, \cdot \rangle_s \), and respective norms \( \| \cdot \|_s \), \( s \geq 0 \). When \( s = 0 \), \( H^0(\Omega) \) is the usual \( L^2(\Omega) \), in which case the norm and inner product will be denoted by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \), respectively. \( L^2_0(\Omega) \) will denote the space of \( L^2(\Omega) \) functions \( p \) such that \( \int_{\Omega} p \, dx = 0 \). The space \( H^1_0(\Omega) \) is the closure of the linear space of infinitely differentiable functions with compact support in \( \Omega \) with respect to the norm \( \| \cdot \|_1 \). From now on, we will omit the superscript \( n \) and \( \Omega \) if the dependence of vector norms on dimension is clear by context. We use \( H^1_0(\Omega) \) to denote the dual spaces of \( H^1_0(\Omega) \) with norm defined by

\[ |\phi|_{-1,0} = \sup_{\psi \in H^1_0(\Omega)} \frac{\langle \phi, \psi \rangle}{\| \psi \|_1}. \]

We will use

\[ |u|_\infty = \sup_{x \in \Omega} |u(x)|. \]

Define the product spaces \( H^1_0(\Omega)^n \) and \( L^2(\Omega)^n \) in the usual way with standard product norms and define a space

\[ Q = \{ q \in L^2(\Omega) : (\| q \|^2 + \mu^2 \| \kappa \beta \cdot \nabla q \|^2)^{\frac{1}{2}} < \infty \}, \]

which is a Hilbert space with norm

\[ \| q \|_Q = (\| q \|^2 + \mu^2 \| \kappa \beta \cdot \nabla q \|^2)^{\frac{1}{2}}. \]

Let

\[ H(div; \Omega) = \{ v \in L^2(\Omega)^n : \nabla \cdot v \in L^2(\Omega) \}, \]

which is a Hilbert space (see [13]) under the norm

\[ \| v \|_{H(div; \Omega)} := (\| v \|^2 + \| \nabla \cdot v \|^2)^{\frac{1}{2}}. \]

Following [10] for Stokes equations, introducing the velocity flux variable \( U = \nabla u \), that is,

\[ U = (U_{11}, U_{12}, U_{13}, \ldots, U_{32}, U_{33})^t = \nabla u, \]

the compressible Stokes equations (1.1), with scaled pressure \( p \), may be written as the following equivalent first-order system:

\[
\begin{aligned}
\mathbf{U} - \nabla u &= 0, \quad \text{in } \Omega, \\
-\nabla \cdot \mathbf{U} + \frac{1}{\rho} \beta \cdot \nabla u + \nabla p &= \frac{1}{\rho} f, \quad \text{in } \Omega, \\
\nabla \cdot u + \mu \kappa \beta \cdot \nabla p &= g, \quad \text{in } \Omega, \\
\mathbf{u} &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

where

\[ \beta \cdot \nabla u = (\beta \cdot \nabla u_1, \ldots, \beta \cdot \nabla u_n)^t. \]
Note that the first and last equations in (2.2) implies
\[
\int_\Omega U \, d\Omega = \int_\Omega \nabla u \, d\Omega = 0.
\]
Hence we may add these two conditions to (2.2), so that an extended equivalent first-order system is given as:
\[
\begin{cases}
U - \nabla u = 0, & \text{in } \Omega, \\
- \nabla \cdot U + \frac{\rho}{\mu} \beta \cdot \nabla u + \nabla p = \frac{1}{\mu} f, & \text{in } \Omega, \\
\nabla \cdot u + \mu \kappa \beta \cdot \nabla p = g, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega, \\
\int_\Omega U \, d\Omega = 0, \\
\int_\Omega \nabla u \, d\Omega = 0.
\end{cases}
\]
(2.3)

Some existence, uniqueness, and regularity results for compressible linearized Stokes equations are discussed in [14], [15], [16], and [17]. It is well known that the solutions \( u \in H^1_0(\Omega) \) and \( p \in L^2_0(\Omega) \) to (1.1) satisfy
\[
\|u\|^2 + \|p\|^2 \leq C(\|f\|^2_{-1,0} + \|g\|^2),
\]
(2.4)
where \( C \) depends on \( \mu, \rho, \kappa \) and \( \beta \).

III. FIRST-ORDER SYSTEM OF LEAST-SQUARES FUNCTIONALS

The main objective in this section is to establish ellipticity and continuity of least-squares functionals based on (2.2) and (2.3) in appropriate Sobolev spaces.

The least-squares functionals considered in this article are
\[
G_1(U, u, p; f, g) = \| - \nabla \cdot U + \frac{\rho}{\mu} \beta \cdot \nabla u + \nabla p - \frac{1}{\mu} f \|_{-1,0}^2 + \| \nabla \cdot u + \mu \kappa \beta \cdot \nabla p - g \|^2 + \| U - \nabla u \|^2.
\]
(3.1)
and
\[
G_2(U, u, p; f, g) = G_1(U, u, p; f, g) + \left( \int_\Omega U \, d\Omega \right)^2 + \left( \int_\Omega \nabla u \, d\Omega \right)^2.
\]
(3.2)

Define
\[
M(U, u, p) = \|U\|^2 + \|u\|^2_1 + \|p\|^2_\beta,
\]
and let
\[
\mathcal{V} := L^2(\Omega)^n \times H^1_0(\Omega)^n \times (Q \cap L^2_0(\Omega)).
\]

The first-order system least-squares variational problem for the compressible Stokes equations corresponding (2.2) and (2.3) is to minimize the quadratic functional \( G_i \) (\( i = 1, 2 \)) over \( \mathcal{V} \): find \( (U, u, p) \in \mathcal{V} \) such that, for \( i = 1, 2 \),
\[
G_i(U, u, p; f, g) = \inf_{(V, v, q) \in \mathcal{V}} G_i(V, v, q; f, g).
\]
(3.3)

The following lemma was proved by Nečas [18] and also can be found in [13].
Lemma 3.1. There is a constant $C_\Omega > 0$, which depends on $\Omega$ such that
\[ \|p\| \leq C_\Omega \|\nabla p\|_{-1}, \text{ for } p \in L_0^2(\Omega). \]

Theorem 3.1. Assume that there are $\mu$ and $\beta$ such that $\frac{1}{\mu} \|\rho \beta\|_\infty + \mu \|\nabla \cdot (\kappa \beta)\|_\infty$ is small enough. Then there are two positive constants $c$ and $C$ such that for all $(U, u, p) \in V$,
\[ cM(U, u, p) \leq G_1(U, u, p; 0, 0) \leq CM(U, u, p). \quad (3.4) \]

Proof. Upper bound in (3.4) is a simple consequence from the triangle inequality, Cauchy-Schwarz inequality and the bound of $\frac{1}{\mu} \|\rho \beta\|_\infty$. Because of limiting arguments, it is enough to show that lower bound in (3.4) holds for $e V = H(div; \Omega)^n \times H_0^1(\Omega)^n \times (L_0^2(\Omega) \cap Q)$. For any $(U, u, p) \in V$ and $\phi \in H_0^1(\Omega)^n$, from Green’s formula and Poincaré-Friedrichs inequality we have
\[
\langle \nabla p, \phi \rangle = \left( -\nabla \cdot U + \frac{\rho}{\mu} (\beta \cdot \nabla) u + \nabla p, \phi \right) - \langle U, \nabla \phi \rangle - \frac{1}{\mu} \langle \rho(\beta \cdot \nabla) u, \phi \rangle
\leq G_1^2(U, u, p; 0, 0) \|\phi\|_1 + \|U\| \|\nabla \phi\| - \frac{1}{\mu} \langle \rho(\beta \cdot \nabla) u, \phi \rangle
\leq G_1^2(U, u, p; 0, 0) \|\phi\|_1 + \|U\| \|\phi\| + C_1 \|\rho \beta\|_\infty \|\nabla u\| \|\phi\|_1,
\]
where $C_1$ is a positive constant. Combining this with Lemma 3.1 yields that
\[ \|p\| \leq C_\Omega \left( G_1^2(U, u, p; 0, 0) + \|U\| + \frac{C_1}{\mu} \|\rho \beta\|_\infty \|\nabla u\| \right). \quad (3.5) \]

Since
\[
\|\nabla u\|^2 = \langle \nabla u - U, \nabla u \rangle + \langle U, \nabla u \rangle
\leq G_1^2(U, u, p; 0, 0) \|\nabla u\| + \|U\| \|\nabla u\|,
\]
we have, by cancelling $\|\nabla u\|$ on both sides,
\[ \|\nabla u\| \leq G_1^2(U, u, p; 0, 0) + \|U\|. \quad (3.6) \]

From (3.5) and (3.6),
\[ \|p\| \leq C_{1,\Omega} \left( G_1^2(U, u, p; 0, 0) + \|U\| \right), \quad (3.7) \]
where $C_{1,\Omega} = C_\Omega \left( 1 + \frac{C_1}{\mu} \|\rho \beta\|_\infty \right)$. 
From the Poincaré-Friedrichs inequality, Green’s formula, (3.6), and (3.7), we have
\[
\begin{align*}
(\mathbf{U}, \mathbf{U}) &= (\mathbf{U} - \nabla \mathbf{u}, \mathbf{U}) + (\nabla \mathbf{u}, \mathbf{U}) \\
&= (\mathbf{U} - \nabla \mathbf{u}, \mathbf{U}) + \left( -\nabla \cdot \mathbf{U} + \frac{1}{\mu} (\rho \mathbf{\beta} \cdot \nabla \mathbf{u}) + \nabla p, \mathbf{u} \right) \\
&\quad - \frac{1}{\mu} (\rho \mathbf{\beta} \cdot \nabla \mathbf{u}, \mathbf{u}) + (p, \nabla \cdot \mathbf{u} + \mu \kappa \mathbf{\beta} \cdot \nabla p) - \mu (p, \kappa \mathbf{\beta} \cdot \nabla p) \\
&\leq G_1^2(\mathbf{U}, \mathbf{u}; p, 0, 0) \|\mathbf{U}\| + C_F G_1^2(\mathbf{U}, \mathbf{u}; p, 0, 0) \|\nabla \mathbf{u}\| \\
&\quad + \frac{C_F}{\mu} \|\mathbf{\rho}\|_\infty \|\nabla \mathbf{u}\|^2 + G_1^2(\mathbf{U}, \mathbf{u}; p, 0, 0) \|p\| + \frac{\mu}{2} \|\nabla \cdot (\kappa \mathbf{\beta})\|_\infty \|\mathbf{U}\|^2,
\end{align*}
\]
where
\[
C_2 = C_F + \frac{2}{\mu} C_F \|\mathbf{\rho}\|_\infty + C_{1,\Omega} + C_{1,\Omega}^2 \mu \|\nabla \cdot (\kappa \mathbf{\beta})\|_\infty, \quad \text{and} \quad C_3 = 1 + C_F + C_{1,\Omega},
\]
and
\[
C_4 = \frac{2}{\mu} C_F \|\mathbf{\rho}\|_\infty + \frac{\mu}{2} C_{1,\Omega}^2 \|\nabla \cdot (\kappa \mathbf{\beta})\|_\infty,
\]
and \((p, (\mathbf{\beta} \cdot \nabla p)) = -\frac{1}{2} \int_{\Omega} (\nabla \cdot \mathbf{\beta}) p^2 \, d\Omega\) is used in the first inequality. Recall the \(\epsilon\)-inequality
\[
2ab \leq \frac{1}{\epsilon} a^2 + b^2.
\]
Now using the assumption that \(\mu\) and \(\mathbf{\beta}\) are chosen so that
\[
\frac{2}{\mu} C_F \|\mathbf{\rho}\|_\infty + \frac{\mu}{2} C_{1,\Omega}^2 \|\nabla \cdot (\kappa \mathbf{\beta})\|_\infty
\]
is small enough, we may have bounds of \(C_2\) and \(C_3\) independent of \(\mu\) and \(\mathbf{\beta}\) and with an absolute constant \(\delta < 1\)
\[
C_4 \leq 1 - \delta,
\]
so that, using \(\epsilon\)-inequality in (3.8), we have
\[
\|\mathbf{U}\|^2 \leq C_5 G_1(\mathbf{U}, \mathbf{u}; p, 0, 0)
\]
(3.9)
where the constant \(C_5\) depends on \(\Omega\). We now then have, using (3.9) in (3.7) and (3.6),
\[
\|p\|^2 \leq C_6 G_1(\mathbf{U}, \mathbf{u}; p, 0, 0)
\]
(3.10)
and
\[
\|\nabla \mathbf{u}\|^2 \leq C_6 G_1(\mathbf{U}, \mathbf{u}; p, 0, 0),
\]
(3.11)
where \(C_6\) is a positive constant dependent on \(\Omega\). Finally, we have
\[
\mu^2 \|\kappa \mathbf{\beta} \cdot \nabla p\|^2 = (\mu \kappa \mathbf{\beta} \cdot \nabla p + \nabla \cdot \mathbf{u}, \mu \kappa \mathbf{\beta} \cdot \nabla p) - (\nabla \cdot \mathbf{u}, \mu \kappa \mathbf{\beta} \cdot \nabla p)
\]
\[
\leq \left( G_1^2(\mathbf{U}, \mathbf{u}; p, 0, 0) + 2 \|\nabla \mathbf{u}\| \right) \mu \|\kappa \mathbf{\beta} \cdot \nabla p\|.
\]
Then, cancelling \( \mu \| \kappa \beta \cdot \nabla p \| \) and squaring the remaining, we have from (3.11),

\[
\mu^2 \| \kappa \beta \cdot \nabla p \|^2 \leq C_6 G_1(U, u, p; 0, 0) \tag{3.12}
\]

The theorem now follows from (3.9), (3.10), (3.12) and the usual continuity arguments.

As a result of this theorem, the following theorem comes easily, from which the existence of the unique minimizer of the functional \( G_2 \) can be shown.

**Theorem 3.2.** Under the same assumption of Theorem 3.1, there are two positive constants \( c \) and \( C \) such that for all \( (U, u, p) \in \mathcal{V} \),

\[
c M(U, u, p) \leq G_2(U, u, p; 0, 0) \leq C M(U, u, p).
\]

### IV. Finite Element Approximations

In this section, we provide the finite element approximation on the minimization of the least-squares functionals \( G_1 \) and \( G_2 \) defined in (3.1) and (3.2), respectively. Let \( T_h \) be a family of triangulations of \( \Omega \) by a standard finite element subdivisions of \( \Omega \) into quasi-uniform triangles with \( h = \max \{ \text{diam}(K) : K \in T_h \} \). Near a curved portion of the boundary \( \partial \Omega \), we use isoparametric triangles with one curved side. It is assumed that the curved elements in the triangulations of the domain of \( \Omega \) not to be “too curved,” so that the standard interpolation error on a straight triangle will hold for a curved triangle (see [19], [20], and [21]). Let \( \mathcal{V}_h := U_h \times S_{0,h} \times \mathcal{W}_h \) be the finite dimensional subspace of \( \mathcal{V} \) with the following properties: there exist a constant \( C \) and integers \( r, s, k, \) for any \( (U, u, p) \in (H^r(\Omega)^n \times H^s(\Omega)^n) \cap \mathcal{V} \), \( 1 \leq r, 1 \leq s \) and \( 1 \leq k \), there exists a pair \( (U_h, u_h, p_h) \in \mathcal{V}_h \) such that

\[
\inf_{U \in U_h} \{ \| U - U_h \| + h \| U - U_h \|_1 \} \leq C h^r \| U \|_r, \tag{4.1}
\]

\[
\inf_{u \in S_{0,h}} \{ \| u - u_h \| + h \| u - u_h \|_1 \} \leq C h^{s+1} \| u \|_{s+1}, \tag{4.2}
\]

\[
\inf_{p \in \mathcal{W}_h} \{ \| p - p_h \| + h \| p - p_h \|_1 \} \leq C h^{k+1} \| p \|_{k+1}. \tag{4.3}
\]

The finite element approximation to (3.3) becomes: find \( (U_h, u_h, p_h) \in \mathcal{V}_h \) that satisfies

\[
G_i(U_h, u_h, p_h; f, g) = \inf_{(V_h, v_h, q_h) \in \mathcal{V}_h} G_i(V_h, v_h, q_h; f, g), \quad (i = 1, 2).
\]

Since the functional \( G_1 \) involves the \( H^{-1} \) norm, we need a solution operator of boundary value problem for its evaluation, for which we adopt the well-known \( H^{-1} \) norm technique proposed in [22].

Let \( T : H_0^{-1}(\Omega)^n \rightarrow H_0^1(\Omega)^n \) be the solution operator \( u = T f \) for the following elliptic boundary value problem:

\[
\left\{ \begin{array}{l}
-\Delta u + u = f, \quad \text{in} \quad \Omega, \\
\quad u = 0, \quad \text{on} \quad \partial \Omega.
\end{array} \right.
\]

The following well-known result can be found in Lemma 2.1 in [22].

\[
(f, T f) = \| f \|^2_{-1,0} = \sup_{\phi \in H_0^{-1}(\Omega)^n} \frac{(f, \phi)^2}{\| \phi \|^2_1} \quad \text{for all} \quad f \in H_0^{-1}(\Omega)^n. \tag{4.4}
\]
Then, from (4.4), we have

\[
G_1(U, u, p; 0, 0) = \left( T(-\nabla \cdot U + \frac{\rho}{\mu} \beta \cdot \nabla u + \nabla p), -\nabla \cdot U + \frac{\rho}{\mu} \beta \cdot \nabla u + \nabla p \right)
+ (\nabla \cdot u + \mu \kappa \beta \cdot \nabla p, \nabla \cdot u + \mu \kappa \beta \cdot \nabla p) + (U - \nabla u, U - \nabla u)
\]

and

\[
G_2(U, u, p; 0, 0) = \left( T(-\nabla \cdot U + \frac{\rho}{\mu} \beta \cdot \nabla u + \nabla p), -\nabla \cdot U + \frac{\rho}{\mu} \beta \cdot \nabla u + \nabla p \right)
+ (\nabla \cdot u + \mu \kappa \beta \cdot \nabla p, \nabla \cdot u + \mu \kappa \beta \cdot \nabla p) + (U - \nabla u, U - \nabla u)
+ \frac{1}{|\Omega|} \left( \int_{\Omega} U \, d\Omega - \int_{\Omega} U \, d\Omega \right) + \frac{1}{|\Omega|} \left( \int_{\Omega} \nabla u \, d\Omega - \int_{\Omega} \nabla u \, d\Omega \right),
\]

where $|\Omega|$ denotes the measure of domain $\Omega$.

**Theorem 4.1.** Suppose that the assumption of Theorem 3.1 holds. Let $(U, u, p) \in V$ be the solution of the minimization of $G_1$ over $\mathcal{V}$ and $(u_h, p_h)$ be the unique minimizer of $G_1$ over $\mathcal{V}_h$. Then

\[
\|U - U_h\|^2 + \|u - u_h\|^2 + \|p - p_h\|^2 
\leq C \inf_{(v_h, q_h) \in \mathcal{V}_h} \left( \|U - V_h\|^2 + \|u - v_h\|^2 + \|p - q_h\|^2 \right).
\]  

(4.5)

**Proof.** For convenience, let

\[
[U, u, p; V, v, q] = \left( T(-\nabla \cdot U + \frac{\rho}{\mu} \beta \cdot \nabla u + \nabla q), -\nabla \cdot V + \frac{\rho}{\mu} \beta \cdot \nabla v + \nabla q \right)
+ (\nabla \cdot u + \mu \kappa \beta \cdot \nabla p, \nabla \cdot v + \mu \kappa \beta \cdot \nabla q) + (U - \nabla u, V - \nabla v).
\]

(4.4), Theorem 3.1, the orthogonality of the error $(U - U_h, u - u_h, p - p_h)$ to $\mathcal{V}_h$, with respect to the above inner product and Cauchy-Schwarz inequality imply that, for all $(V_h, v_h, q_h) \in \mathcal{V}_h$,

\[
\|U - U_h\|^2 + \|u - u_h\|^2 + \|p - p_h\|^2
\leq C \left[ \|U - U_h, u - u_h, p - p_h\| \cdot \|U - U_h, u - u_h, p - p_h\| \right]
\leq C \left[ \|U - U_h\|^2 + \|u - u_h\|^2 + \|p - p_h\|^2 \right]
\leq C \left[ \|U - V_h\|^2 + \|u - v_h\|^2 + \|p - q_h\|^2 \right],
\]

where $C$ depends on $\Omega$. Note that the last inequality holds because $T$ is positive definite and symmetric. Canceling the common factor on both sides and squaring the remains, we have the conclusion.
Corollary 4.1. Suppose that $\mu$ satisfies the same assumption of Theorem 3.1. Assume that 
\((U, u, p) \in (H^r(\Omega))^n \times H_0^1(\Omega)^n \times H^{k+1}(\Omega)) \cap V\) is the solution of the minimization problem for \(G_1\) in (3.3) and \((U_h, u_h, p_h)\) is the unique minimizer of \(G_1\) over \(V_h\). Then 
\[ \| U - U_h \|^2 + \| u - u_h \|^2 + \| p - p_h \|^2 \leq C \left( h^{2r} \| U \|^2 + h^{2k} \| u \|^2_{k+1} + h^{2k} \| p \|^2_{k+1} \right), \]
where \(C\) depends on \(\Omega\).

Proof. By (4.1),(4.2),(4.3) and Theorem 4.1, we can get the above result.

Note that one may get easily a similar approximation result for minimizing \(G_2\) functional as Corollary 4.1. We notice that the inequality (4.5) can be expected to give an optimal rate of convergence. As any choice of subspaces \(U_h, S_{0,h}\) and \(W_h\), (4.6) shows that the error bounds for velocity flux and velocity obtained here are the best approximations in the subspaces \(U_h\) and \(S_{0,h}\), but one may see that the \(L^2\) error bound for pressure is one order less than the best approximation in the subspace \(W_h\). It is noted in [1] that when the mixed finite element methods are applied to (1.1), taking the so called MINI elements for velocity and pressure which are subject to the inf-sup condition, the optimal convergence rate corresponding to the error bound (4.5) may not be obtained.

In order to evaluate functionals \(G_1\) and \(G_2\) on finite dimensional subspaces, it is required to replace the operator \(T\) by a computable and equivalent operator \(T_h\). Let \(T_h\) on the finite dimensional subspace \(S_{0,h}\) of \(H^1_0(\Omega)^n\) be the discrete Galerkin solution operator corresponding to \(T\). Since a slight modification on arguments for the functional \(G_1\) can be applied to the functional \(G_2\), we mainly focus on evaluation of the functional \(G_2\) from now on. Assume that there is a good pre-conditioner \(B_h: S_{0,h} \rightarrow S_{0,h}\), which is symmetric, positive definite with respect to \(L^2\) inner product and equivalent to \(T_h\), that is,
\[ c(T_h v, v) \leq (B_h v, v) \leq C(T_h v, v), \quad \text{for all } v \in S_{0,h}. \]

Define 
\[ T_h = h^2 I + B_h, \]
where \(I\) denotes the identity operator and 
\[ G_h(U, u, p; \frac{1}{\mu} f, g) \]
\[ = \left( T_h (-\nabla \cdot U + \frac{\rho}{\mu} \beta \cdot \nabla u + \nabla p - \frac{1}{\mu} f), \quad -\nabla \cdot U + \frac{\rho}{\mu} \beta \cdot \nabla u + \nabla p - \frac{1}{\mu} f \right) \]
\[ + (\nabla \cdot u + \mu \kappa \beta \cdot \nabla p - g, \quad \nabla \cdot u + \mu \kappa \beta \cdot \nabla p - g) + (U - \nabla u, U - \nabla u). \]

Let \(Q_h\) be the \(L^2(\Omega)^n\) orthogonal projection operator onto \(S_{0,h}\). We further assume that 
\[ \| Q_h u \|_1 \leq C \| u \|_1 \quad \text{for all } u \in H^1_0(\Omega)^n. \]

The symmetric properties of operators \(B_h\) and \(T_h\) with respect to \(L^2\) inner product yield \(B_h = B_h Q_h\) and \(T_h = T_h Q_h\). Thus (4.7) holds for any \(v \in L^2(\Omega)^n\). Hence, using (4.8) and the definition of discrete solution operator \(T_h\) (refer to (4.4)), it follows that 
\[ c \| Q_h u \|_{2,1,0}^2 \leq (u, T_h u) \leq C \| u \|_{2,1,0}^2 \quad \text{for all } u \in H^1_0(\Omega)^n. \]

It is easy to check that, from the approximation result (4.2) with \(s = 1\) and Cauchy-Schwarz inequality,
\[ \| v - Q_h v \|_{1,0} \leq C h \| v \|, \quad \text{for all } v \in L^2(\Omega)^n, \]
so that, (4.7), (4.9) and (4.10) imply that
\[ \|f\|_{1,0} \leq C(f, T_h f) \quad \text{for all } \ f \in L^2(\Omega)^n. \quad (4.11) \]

**Lemma 4.1.** Suppose that the assumption of Theorem 3.1 holds and assume that \( V_h \) satisfies the inverse inequalities such that \( h \|U_h\|_1 \leq C\|U_h\|, \ h \|u_h\|_1 \leq C\|u_h\|, \) and \( h \|
abla p_h\| \leq C\|p_h\|.

For any \((U_h, u_h, p_h) \in V_h\), there are two positive constants \( c \) and \( C \) independent of \( h \) such that
\[ c(\|U_h\|_1^2 + \|u_h\|_1^2 + \|p_h\|_0^2) \leq G_h(U_h, u_h, p_h; 0, 0) \quad (4.12) \]
and
\[ G_h(U_h, u_h, p_h; 0, 0) \leq C(\|U_h\|_1^2 + \|u_h\|_1^2 + \|p_h\|_0^2). \quad (4.13) \]

**Proof.** Since \( T_h = h^2 I + B_h \), using (4.7) and definitions of \( T_h \) (refer to (4.4)), we have
\[ (T_h \mathbf{v}, \mathbf{v}) \leq C(h^2 \|\mathbf{v}\|_1^2 + \|\mathbf{v}\|_{1,0}^2) \quad \text{for all } \mathbf{v} \in L^2(\Omega)^n. \quad (4.14) \]

Then (4.14), triangle inequality and inverse inequality imply that
\begin{align*}
& (T_h (-\nabla \cdot U_h + \frac{\mu}{\rho} \beta \cdot \nabla u_h + \nabla p_h), -\nabla \cdot U_h + \frac{\mu}{\rho} \beta \cdot \nabla u_h + \nabla p_h) \\
& \leq C(\|U_h\|_1^2 + (\frac{1}{\rho})^2 \|\rho \beta\|_\infty^2 \|u_h\|_1^2 + \|p_h\|_0^2 + \|\nabla \cdot U_h + \frac{\mu}{\rho} \beta \cdot \nabla u_h + \nabla p\|_{1,0}^2) \\
& \leq C(\|U_h\|_1^2 + \|u_h\|_1^2 + \|p_h\|_0^2 + \|\nabla \cdot U_h + \frac{\mu}{\rho} \beta \cdot \nabla u_h + \nabla p\|_{1,0}^2),
\end{align*}

where the bound of \( \frac{1}{\rho} \|\rho \beta\|_\infty \) is used. This argument completes the upper bound (4.13). The lower bound (4.12) can be proved easily by (4.11) and Theorem 3.1.

Because of this lemma, the similar proofs of Theorem 4.1 and Lemma 4.1 lead to:

**Theorem 4.2.** Suppose that \( \mu \) satisfies the same assumption of Theorem 3.1. Assume that \((U, u, p) \in V\) is the solution of the problem (3.3). Let \((U_h, u_h, p_h) \in V_h\) be the unique minimizer of \( G_h \). Then
\[ \|U - U_h\|^2 + \|u - u_h\|^2 + \|p - p_h\|^2 \leq C \inf_{(V_h, u_h, q_h) \in V_h} (\|U - V_h\|^2 + \|u - v_h\|^2 + \|p - q_h\|^2), \quad (4.15) \]
where \( C \) depends on \( \Omega \).

**Corollary 4.2.** Suppose that \( \mu \) satisfies the same assumption of Theorem 3.1. Assume that \((U, u, p) \in H^{r}(\Omega)^n \times H_{0}^{r+1}(\Omega)^n \times H^{r+1}(\Omega) \) \( \cap V \) is the solution of the problem (3.3). Let \((U_h, u_h, p_h) \in V_h\) be the unique minimizer of \( G_h \). Then
\[ \|U - U_h\|^2 + \|u - u_h\|^2 + \|p - p_h\|^2 \leq C (h^{2r} \|U\|_r^2 + h^{2k} \|u\|_{r+1}^2 + h^{2k} \|p\|_{r+1}^2), \quad (4.16) \]
where \( C \) depends on \( \Omega \).

We briefly note that the inequality (4.15) can also give an optimal rate of convergence and the inequality (4.16) shows that the error bounds for velocity flux and velocity are the best approximations on the subspaces \( U_h \) and \( S_h \).

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References